Effect of shaft misalignment and friction force on time varying mesh stiffness of spur gear pair

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ARTICLE INFO

Article history:
Received 29 August 2014
Received in revised form 24 December 2014
Accepted 31 December 2014
Available online 12 January 2015

Keywords:
Misalignment
Sliding friction
Mesh stiffness
Cracked tooth

ABSTRACT

Shaft misalignment and sliding friction between meshing teeth are considered as primary excitation to generate vibrations and extra dynamic loads on transmitting gear teeth. Time varying mesh stiffness (TVMS) is an important parameter to understand the dynamics of meshing gear pair. Potential energy method is one of the most acceptable methods to calculate TVMS. This paper proposes a computer simulation based approach to study the effect of shaft misalignment and friction on total effective mesh stiffness for spur gear pair. The results showed clearly that misalignment and friction affect TVMS of gear pair. The effect of misalignment and friction has also been studied for cracked gear pair and results are discussed.

1. Introduction

Gear system is one of the most common transmission systems used to transmit motion and power in various industries. In major industries condition monitoring and fault diagnosis of gear system is an important task. The vibration response of gear system plays an important role in fault diagnosis and condition monitoring.

Vibration response of gear pair is closely related to TVMS of the gear pair. Two methods are being widely used by researchers to calculate TVMS of gear pair, viz. finite element method (FEM) and analytical method (AM). Many researchers have proposed their analytical methods to calculate the TVMS of healthy and damaged gears. Yang and Sun [1] in 1985 proposed a value of Hertzian energy which is further extended by Yang and Lin [2] in 1987 to calculate TVMS of a gear pair by the potential energy method by including bending energy and axial compressive energy with Hertzian energy. This model was further refined by Tian [3] in 2004 by taking the shear energy into consideration. Tian [3] also discussed the effect of chipped tooth, cracked tooth and a broken tooth. Wu [4] presented the refined model of Tian [3] for faulty gear pair for calculating the total effective mesh stiffness as a function of crack length, crack intersection angle and rotation angle of gear for a pair of meshing spur gears consisting of a perfect gear and a pinion with cracked tooth. Wu et al. [5] studied the effect of crack growth in gear tooth on total mesh stiffness. They have also done dynamic modeling to simulate vibration response of meshing gear pair. Fillet foundation deflection proposed by Sainso et al. [6] in 2003 is used by Chari et al. [7] in 2009 to develop a model to calculate the TVMS using potential energy method.

Chen and Shao [8,9] proposed an analytical mesh stiffness model of spur gear with tooth root crack propagating along both tooth width and crack depth. They have also included the effect of the gear tooth errors. Influences of the tooth profile...

Investigation of gear mesh stiffness has been carried out extensively for healthy and cracked gears. But very less work is done on misalignment errors of spur gear pair. Ameen [20] in 2010 studied various effects of shaft misalignment on the stress distribution of spur gear using FEM technique. Recently Li [21] used FEM technique to study various parameters including the effects of misalignment errors on mesh stiffness of spur gear pair. Results show that due to misalignment errors mesh stiffness of spur gear pair decreases. In this paper TVMS has been calculated for spur gear pair mounted on misaligned shaft using potential energy method. Effect of friction force on TVMS for misaligned shaft has also been explored.

2. Potential energy model for time varying mesh stiffness calculation for spur gear pair

2.1. Mesh stiffness calculations for healthy spur gear pair

The gear mesh stiffness model used in this study is based on potential energy method proposed by Yang and Lin [2] in 1987 which is further refined by Tian [3] in 2004. The energy stored in meshing gear system was assumed to include four components out of which three components; Hertzian energy, bending energy and axial compressive energy is given in [2]; and fourth component; shear energy was proposed in [3]. Thus, for the single – tooth contact, the total effective TVMS can be expressed as [3],

\[
k_i = \frac{1}{k_h + 1/k_s + 1/k_a + 1/k_{a1} + 1/k_{a2} + 1/k_{a3}}
\]

where \(k_h, k_b, k_s\) and \(k_a\) represents the Hertzian, bending, shear and axial compressive mesh stiffness, respectively and subscripts 1 and 2 denote the driving and driven gears respectively. For the double-tooth pair contact, the total effective TVMS is the sum of the two pair’s stiffness, which can be expressed as [3],

\[
k_i = \frac{2}{\sum_{i=1}^{2} 1/k_{h,i} + 1/k_{b,i} + 1/k_{s,i} + 1/k_{a1,i} + 1/k_{a2,i} + 1/k_{a3,i}}
\]

where \(i = 1\) represents the first pair of meshing teeth and \(i = 2\) represents the second pair of meshing teeth.

2.2. Mesh stiffness of gear pair with a cracked pinion

If crack has been initiated at the root of a single tooth of a pinion then the above formula is not valid because bending stiffness and shear stiffness will change due to influence of the crack. This phenomenon will occur because when the crack is present, the effective area moment of inertia and the area of the cross section will change [3]. So, for single tooth mesh period the total effective mesh stiffness is given by,

\[
k_{i,\text{crack}} = \frac{1}{k_h + 1/k_{b,\text{crack}} + 1/k_{s,\text{crack}} + 1/k_{a1} + 1/k_{a2} + 1/k_{a3}}
\]

where \(k_{b,\text{crack}}\) and \(k_{s,\text{crack}}\) are the bending mesh stiffness and shear mesh stiffness of cracked tooth respectively. And for double tooth mesh period the total effective mesh stiffness can be written as,

\[
k_{i,\text{crack}} = \frac{2}{\sum_{i=1}^{2} 1/k_{h,i} + 1/k_{b,i} + 1/k_{s,i} + 1/k_{a1,i} + 1/k_{a2,i} + 1/k_{a3,i}}
\]

3. Proposed potential energy model to calculate TVMS of misaligned gear shaft including friction

3.1. Effect of shaft misalignment on TVMS of spur gear pair

Shaft misalignment is considered as one of the most common problems in rotating machines, which leads to generate vibrations and extra dynamic loads on transmitting gear teeth. Good alignment for gear shaft means that shafts are parallel
to each other, do not have angular misalignment and are not offset with respect to their centerlines. Fig. 1 shows the difference between aligned shaft and misaligned shaft with $\theta$ of misalignment angle.

According to the properties of involute profile, the line of action is the common normal to the tooth profile. Correspondingly, the acting force $F$ of the contact teeth should be always along the line of action. But when there is shaft misalignment in gear shaft the force $F$ will act making an angle $\theta$ with normal. This effect will change the value of forces acting on gear tooth. This concept is explained in Fig. 2, where due to misalignment, value of radial and tangential forces change. Pressure angle ($a_1$) will remain same in both the cases.

On resolving the forces as shown in Fig. 2 for aligned shaft and misaligned shaft the values of forces $F_t$, $F_r$ and $F_a$ can be obtained, where $F_t$, $F_r$ and $F_a$ are tangential, radial and axial components of force $F$ respectively on the gear tooth. Expressions for various forces acting on gear tooth are mentioned in Table 1.

The Hertzian stiffness and the axial compressive stiffness do not change due to misalignment, and can be given as

$$k_h = \frac{\pi El}{4(1-v^2)}$$

$$1 = \int_{-x_1}^{x_2} \frac{(x_2-x) \cos x \sin^2 x_1}{2El[\sin x + (x_2-x) \cos x]} \, dx$$

The bending mesh stiffness and the shear mesh stiffness vary due to shaft misalignment. The potential bending energy and shearing energy stored in the meshing pair can be expressed as

$$U_b = \int_0^d \frac{[F_t(d-x) - F_h]^2}{2EI_x} \, dx = \frac{F^2}{2k_b}$$

$$U_s = \int_0^d 1.2F^2 \frac{2EI_x}{2CA_x} \, dx = \frac{F^2}{2k_s}$$

where $k_b$, $k_s$, $k_a$, and $k_s$ represent Hertzian mesh stiffness, bending mesh stiffness, axial compressive mesh stiffness, and shear mesh stiffness, respectively. Definitions of $d$, $x$, and $h$ are shown in Fig. 3 and values of $G$, $I_x$, and $A_x$ can be calculated as

$$G = \frac{E}{2(1+v)}; \quad A_x = 2h_3L; \quad I_x = \frac{2}{3} h_3^3L.$$  

Using these parameters and putting their values in above equations the expression for bending mesh stiffness and the shear mesh stiffness can be derived for misaligned shaft as;

$$1 = \int_{-x_1}^{x_2} \frac{3 \cos x_1 \cos \theta ([x_2-x] \sin x - \cos x) - 2 \sin^2 \theta/2([x_1 + x_2] \sin x_1 \cos x_1) + \cos^3 x_1 \cos \theta + \sin^2 x_1}{2EI[\sin x + (x_2-x) \cos x]} \, dx$$

$$1 = \int_{-x_1}^{x_2} \frac{1.2(1+v)(x_2-x) \cos x \cos^2 x_1 \cos^2 \theta}{EI[\sin x + (x_2-x) \cos x]} \, dx$$

Apart from these variations, due to misalignment load distribution on gear tooth will also change and it will add one more stiffness component.

3.1.1. Effect of load distribution due to shaft misalignment

Load distribution on gear tooth for aligned shaft is assumed to be uniformly distributed, but for the misaligned shaft it is not the same. Ameen [20] in 2010 proposed that due to misalignment error load distribution will follow a parabolic distri-
bution. Using this concept a load distribution curve for aligned and misaligned shaft can be drawn as shown in Fig. 4. In Fig. 4 load distribution of tangential force is shown.

In Fig. 4(a) aligned shaft case is shown in which load distribution is uniform and the total load acts at its centre of gravity. For misaligned shaft case, load distribution is parabolic and the resultant load acts at its centre of gravity, which is located at $3/8$th of length of contact from maximum load position as shown in Fig. 4(b). This resultant tangential force will try to twist the gear tooth with a torque $T$ as shown in Fig. 4(b). A new term is introduced to consider this phenomenon while calculating TVMS of the gear pair and is named as torsion stiffness ($k_s$).

In this study it is assumed that mating teeth are two isotropic elastic bodies and the torsion effect on gear tooth is only because of tangential component of the acting force as shown in Fig. 4(b).

The torsion energy stored in a tooth can be expressed as

$$ U_T = \frac{F^2}{2k_s} $$

where $k_s$ shows the effective torsion stiffness.

---

**Table 1**

Expression for various forces acting on gear tooth.

<table>
<thead>
<tr>
<th>Force</th>
<th>For aligned shaft</th>
<th>For misaligned shaft</th>
</tr>
</thead>
<tbody>
<tr>
<td>Tangential ($F_t$)</td>
<td>$F \cos \alpha_1$</td>
<td>$F \cos \alpha_1 \cos \theta$</td>
</tr>
<tr>
<td>Radial ($F_r$)</td>
<td>$F \sin \alpha_1$</td>
<td>$F \sin \alpha_1$</td>
</tr>
<tr>
<td>Axial ($F_a$)</td>
<td>$0$</td>
<td>$F \cos \alpha_1 \sin \theta$</td>
</tr>
</tbody>
</table>
The potential energy stored due to torque can be obtained by (refer to Eq. (5.26) in Ref. [22]):

\[ U_s = \int_0^d \frac{T^2}{2 GI_{px}} \, dx \]

where \( I_{px} \) represents polar moment of inertia of the section where the distance from the tooth root is \( x \). It can be calculated as

\[ I_{px} = \frac{2 h_c L (L^2 + 4 h_c^2)}{12} \]

And torque \( T \) represents the torsion effect of total tangential force ‘\( F_t \)’ on gear tooth. It can be expressed as

\[ T = F_t \times t \]

where \( t \) is the distance between point at which load is assumed to be acting (i.e. at C.G.) and mid-point of the line of contact as shown in Fig. 4(b). Its value can be expressed as

\[ t = \frac{L}{2} - \frac{3L}{8}; \quad \Rightarrow t = \frac{L}{8} \]

Definition and values of \( G, L, h_c \) can be obtained from Ref. [4].

To investigate the parameter properties at some angular displacement of the pinion/gear; relationship are expressed as angular variables instead of linear variables.

Using the values given in Ref. [3] for \( L, d, h_c, dx \) and substituting the values in Eqs. (11)–(15)

\[ \frac{1}{k_t} = \int_{-\pi}^{\pi} \frac{3L(1 + \nu)(h_c - \alpha) \cos^2 \theta \cos \alpha}{16E \sin \alpha + (h_c - \alpha) \cos \alpha} \left( 4R_a^2 \sin \alpha + (h_c - \alpha) \cos \alpha \right)^2 + \frac{L^2}{3} \, d\alpha \]

3.1.2. Effect of misaligned shaft with cracked tooth on TVMS

In this study, a crack at tooth root with crack depth \( q \) along the width of the tooth has been considered as shown in Fig. 3. Crack length \( q \) is less than half of the base chordal thickness. The intersection angle \( \nu \) between crack and the central line of tooth is constant. Previous research shows that axial compressive stiffness is not affected by occurrence of crack [3]. The change in gear stiffness will occur due to change in bending and shear stiffness. Using (9) and (10) the required expression for bending mesh stiffness and shear mesh stiffness considering misalignment for cracked tooth gear can be given as;

\[ \frac{1}{k_{t,\text{crack}}} = \int_{-\pi}^{\pi} \frac{12 \cos \alpha \cos \theta (h_c - \alpha) \sin \alpha - \cos \alpha - 2 \sin^2 \theta / 2 \left( (\alpha_1 + \alpha_2) \sin \alpha \cos \alpha_1 + \cos^2 \alpha_1 \cos \theta + \sin^2 \alpha_1 \right)^2 (h_c - \alpha) \cos \alpha}{EL \sin \alpha_2 - \frac{h_c}{\sin \alpha_1} \sin \nu + \sin \alpha + (h_c - \alpha) \cos \alpha} \, d\alpha \]

\[ \frac{1}{k_{s,\text{crack}}} = \int_{-\pi}^{\pi} \frac{2.4(1 + \nu)(h_c - \alpha) \cos \alpha \cos^2 \alpha_1 \cos^2 \theta \sin \alpha_2 - \frac{h_c}{\sin \alpha_1} \sin \nu + \sin \alpha + (h_c - \alpha) \cos \alpha}{EL \sin \alpha_2 - \frac{h_c}{\sin \alpha_1} \sin \nu + \sin \alpha + (h_c - \alpha) \cos \alpha} \, d\alpha \]

3.1.2.1. Effect of crack on torsion stiffness of gear pair. Due to crack, value of \( I_{px} \) will change to \( I_{pxc} \)

\[ I_{pxc} = \begin{cases} \frac{1}{12} (h_c + h_c) L \left( (h_c + h_c)^2 + L^2 \right) & \text{if } x \leq g_c \\ \frac{2 h_c L (L^2 + 4 h_c^2)}{12} & \text{if } x > g_c \end{cases} \]
where \( h_c \) is the distance from the root of the crack to the centre line of the tooth, which corresponds to point G on the tooth profile as shown in Fig. 3(b) and can be calculated by \[ h_c = R_{01} \sin \alpha_2 - q \sin \nu \tag{20} \]

Different gear parameters like \( R_{01}, q, \nu, g_c, h_c, h_s \) are also shown in Fig. 3(b).

Using values obtained in Eqs. (19) and (20) and replacing \( l_{psc} \) with \( l_{pse} \) in Eqs. (12), and (16) will become,

\[
\frac{1}{k_{l,crack}} = \int_{-\alpha_t}^{\alpha_t} \frac{3\left(1 + \nu\right)(\sin \alpha_2 + (\sin \alpha_2 - \sin \alpha) - q/R_{03} \sin \nu)}{8E\left(\sin \alpha_2 + (\sin \alpha_2 - \sin \alpha) - q/R_{03} \sin \nu\right)} \cos \alpha \\sin \alpha d\alpha
\]

Due to addition of torsion stiffness component, the total TVMS as given in Eqs. (1)–(4) will modify as

For healthy spur gear pair;

\[ k_i = \begin{cases} 
\frac{1}{l_{11} + 1/l_{10} + 1/k_{11} + 1/k_{10} + 1/k_{12} + 1/k_{12}} ; & \text{For single tooth mesh period} \\
\frac{1}{l_{21} + 1/l_{20} + 1/k_{21} + 1/k_{20} + 1/k_{22} + 1/k_{22}} ; & \text{For double tooth mesh period} 
\end{cases} \]

where subscripts 1 and 2 denote the driving and driven gears respectively, \( i = 1 \) represents the first pair of meshing teeth, \( i = 2 \) stands for the second pair of meshing teeth.

For cracked pinion gear pair;

\[ k_{l,crack} = \begin{cases} 
\frac{1}{l_{11} + 1/l_{10} + 1/k_{11} + 1/k_{10} + 1/k_{12} + 1/k_{12}} + \frac{1}{l_{21} + 1/l_{20} + 1/k_{21} + 1/k_{20} + 1/k_{22} + 1/k_{22}} ; & \text{For single tooth mesh period} \\
\frac{1}{l_{11} + 1/l_{10} + 1/k_{11} + 1/k_{10} + 1/k_{12} + 1/k_{12}} + \frac{1}{l_{21} + 1/l_{20} + 1/k_{21} + 1/k_{20} + 1/k_{22} + 1/k_{22}} ; & \text{For double tooth mesh period} 
\end{cases} \]

3.2. Effect of friction on TVMS

Various researchers have studied the effect of friction on gear tooth meshing but the use of time varying mesh stiffness for friction studies is not been performed by researchers. For the purpose of explanation, the coefficient of friction (\( \mu \)) is represented as an idealized mathematical entity. The normal contact load is assumed to be equally distributed amongst all the teeth in contact. With the aim of highlighting the effect of friction on TVMS an arbitrary value of \( \mu = 0.1 \) has been chosen for the friction coefficient which can be considered too high for lubricated gears. Nevertheless, it is chosen for demonstration purpose in order to enhance the effect of friction.

The effect of friction force which acts perpendicular to the normal force cannot be completely ignored. During gear meshing, the gear and pinion undergo a rolling and sliding action, except at the pitch point, where pure rolling takes place. Considering friction, the gear tooth contact process can be divided into two phases; start of engagement to pitch point known as approach process and pitch point to disengagement known as recess process. Fig. 4 shows the direction of forces including approach process and pitch point to disengagement known as recess process. Fig. 4 shows the direction of forces including flexure force. In Fig. 4(a) \( F \) is a force acting on gear tooth surface along line of action (LOA), \( F_h \) and \( F_v \) are vertical and horizontal components of acting force \( F \) respectively. Similarly in Fig. 4(b) and (c) \( f \) is friction force which acts perpendicular to force \( F \), \( f_h \) and \( f_v \) are vertical and horizontal components of friction force \( f \) respectively. Pressure angle is represented by \( \alpha_1 \) in Fig. 4.

In Fig. 4(b) and (c) it has been shown that friction force \( f \) always acts perpendicular to the force \( F \), and its value is equal to the product of coefficient of friction \( \mu \) and force \( F \), i.e.

\[ f = \mu F \tag{21} \]

Using (21) and resolving force \( F \) and friction force \( f \) as shown in Fig. 5 modified value of tangential and radial forces acting on gear tooth can be calculated. Table 2 shows the expression for radial and tangential forces acting on gear tooth.

3.2.1. Effect of friction on TVMS of healthy gear pair

Friction force changes the value of tangential force and radial force, due to which bending stiffness and axial compressive stiffness will change. Hertzian stiffness will remain unchanged. Putting values of radial and tangential component in (6)–(8), expression for bending mesh stiffness, shear mesh stiffness and axial compressive stiffness can be given as follows:

Bending stiffness;

\[
\frac{1}{k_b} = \int_{-\alpha_t}^{\alpha_t} \frac{3(1 + \nu)(\sin \alpha_2 + (\sin \alpha_2 - \sin \alpha) - q/R_{03} \sin \nu)}{8E(\sin \alpha_2 + (\sin \alpha_2 - \sin \alpha) - q/R_{03} \sin \nu)} \cos \alpha \sin \alpha d\alpha \tag{22}
\]

Shear stiffness;

\[
\frac{1}{k_s} = \int_{-\alpha_t}^{\alpha_t} \frac{3(1 + \nu)(\sin \alpha_2 + (\sin \alpha_2 - \sin \alpha) - q/R_{03} \sin \nu)}{8E(\sin \alpha_2 + (\sin \alpha_2 - \sin \alpha) - q/R_{03} \sin \nu)} \cos \alpha \sin \alpha d\alpha \tag{23}
\]
Values of radial and tangential forces on gear tooth.

<table>
<thead>
<tr>
<th>Force</th>
<th>No friction</th>
<th>Approach process</th>
<th>Recess process</th>
</tr>
</thead>
<tbody>
<tr>
<td>Radial component ($F_r$)</td>
<td>$F_h$</td>
<td>$F_h + f_h$</td>
<td>$F_h - f_h$</td>
</tr>
<tr>
<td>Tangential component ($F_t$)</td>
<td>$F_v$</td>
<td>$F_v - f_v$</td>
<td>$F_v + f_v$</td>
</tr>
</tbody>
</table>

Where $F_h = F \sin \alpha_1$, $F_v = F \cos \alpha_1$, $f_h = f \cos \alpha_1$, $f_v = f \sin \alpha_1 = \mu F \sin \alpha_1$.

\[
1 \frac{k_s}{k_s} = \int_{-\pi/2}^{\pi/2} \frac{1}{2E}\frac{1}{L} \left\{ 2 \left[ \frac{1 + \cos (\alpha_1)}{2} \right] \sin \alpha_1 \cos \alpha_1 + \frac{1}{2} \sin \alpha_1 \cos \alpha_1 \right\} \cos \alpha_1 d\alpha, \quad (24)
\]

\[
1 \frac{k_s}{k_s} = \int_{-\pi/2}^{\pi/2} \frac{1}{2E}\frac{1}{L} \left\{ 2 \left[ \frac{1 + \cos (\alpha_1)}{2} \right] \sin \alpha_1 \cos \alpha_1 + \frac{1}{2} \sin \alpha_1 \cos \alpha_1 \right\} \cos \alpha_1 d\alpha, \quad (25)
\]

Axial compressive stiffness:

\[
1 \frac{k_a}{k_a} = \int_{-\pi/2}^{\pi/2} \frac{1}{2E}\frac{1}{L} \left\{ 2 \left[ \frac{1 + \cos (\alpha_1)}{2} \right] \sin \alpha_1 \cos \alpha_1 + \frac{1}{2} \sin \alpha_1 \cos \alpha_1 \right\} \cos \alpha_1 d\alpha, \quad (26)
\]

\[
1 \frac{k_a}{k_a} = \int_{-\pi/2}^{\pi/2} \frac{1}{2E}\frac{1}{L} \left\{ 2 \left[ \frac{1 + \cos (\alpha_1)}{2} \right] \sin \alpha_1 \cos \alpha_1 + \frac{1}{2} \sin \alpha_1 \cos \alpha_1 \right\} \cos \alpha_1 d\alpha, \quad (27)
\]

In the above expressions Eqs. (22), (24) and (26) are derived for approach process and Eqs. (23), (25) and (27) are derived for recess process.

3.2.2. Effect of friction on TVMS of cracked spur gear tooth

Considering friction force for the cracked spur gear tooth, values of bending stiffness and shear stiffness will change as mentioned in (28–31). Axial compressive stiffness will remain same and its values are given in (26) and (27).

Bending stiffness:

\[
1 \frac{k_{b,\text{crack}}}{k_{b,\text{crack}}} = \int_{-\pi/2}^{\pi/2} \frac{1}{2E}\frac{1}{L} \left\{ 2 \left[ \frac{1 + \cos (\alpha_1)}{2} \right] \sin \alpha_1 \cos \alpha_1 + \frac{1}{2} \sin \alpha_1 \cos \alpha_1 \right\} \cos \alpha_1 d\alpha, \quad (28)
\]

\[
1 \frac{k_{b,\text{crack}}}{k_{b,\text{crack}}} = \int_{-\pi/2}^{\pi/2} \frac{1}{2E}\frac{1}{L} \left\{ 2 \left[ \frac{1 + \cos (\alpha_1)}{2} \right] \sin \alpha_1 \cos \alpha_1 + \frac{1}{2} \sin \alpha_1 \cos \alpha_1 \right\} \cos \alpha_1 d\alpha, \quad (29)
\]

Shear stiffness:

\[
1 \frac{k_{s,\text{crack}}}{k_{s,\text{crack}}} = \int_{-\pi/2}^{\pi/2} \frac{1}{2E}\frac{1}{L} \left\{ 2 \left[ \frac{1 + \cos (\alpha_1)}{2} \right] \sin \alpha_1 \cos \alpha_1 + \frac{1}{2} \sin \alpha_1 \cos \alpha_1 \right\} \cos \alpha_1 d\alpha, \quad (30)
\]

\[
1 \frac{k_{s,\text{crack}}}{k_{s,\text{crack}}} = \int_{-\pi/2}^{\pi/2} \frac{1}{2E}\frac{1}{L} \left\{ 2 \left[ \frac{1 + \cos (\alpha_1)}{2} \right] \sin \alpha_1 \cos \alpha_1 + \frac{1}{2} \sin \alpha_1 \cos \alpha_1 \right\} \cos \alpha_1 d\alpha, \quad (31)
\]

In the above expressions equation (28) and (30) are derived for approach process and equation (29) and (31) are derived for recess process.
3.3. Effect of misalignment and friction forces on TVMS

Considering both the effects, i.e. friction and misalignment values of gear mesh stiffness components can be given as follows:

3.3.1. Effect of both misalignment and friction on TVMS of healthy spur gear tooth

Bending stiffness:

\[
\frac{1}{k_b} = \int_{\alpha_1}^{\alpha_2} \frac{3}{\sin \alpha} \left\{ \left[ 1 + \cos \alpha \cos \theta \left( \alpha_2 - \alpha \right) \sin \alpha - \cos \alpha \right] - \mu \left( x_1 + \alpha_2 + \sin \alpha \left\{ \left( \alpha_2 - \alpha \right) \sin \alpha - \cos \alpha \right\} \right) \right\}^2 \frac{dx}{(\alpha_2 - \alpha) \sin \alpha}, \quad (33)
\]

\[
\frac{1}{k_s} = \int_{\alpha_1}^{\alpha_2} \frac{3}{(\alpha_2 - \alpha) \sin \alpha} \cos \alpha \sin \alpha \left\{ \left( \alpha_2 - \alpha \right) \sin \alpha - \cos \alpha \right\} \right\}^2 \frac{dx}{(\alpha_2 - \alpha) \sin \alpha}, \quad (34)
\]

Shear stiffness;

\[
\frac{1}{k_s} = \int_{\alpha_1}^{\alpha_2} \frac{2 \left[ 1 + \cos \alpha \cos \theta \left( \alpha_2 - \alpha \right) \sin \alpha - \cos \alpha \right] \cos \alpha \sin \alpha \right\}^2 \frac{dx}{(\alpha_2 - \alpha) \sin \alpha}, \quad (35)
\]

Torsion stiffness;

\[
\frac{1}{k_t} = \int_{\alpha_1}^{\alpha_2} \frac{2 \left[ 1 + \cos \alpha \cos \theta \left( \alpha_2 - \alpha \right) \sin \alpha - \cos \alpha \right] \cos \alpha \sin \alpha \right\}^2 \frac{dx}{(\alpha_2 - \alpha) \sin \alpha}, \quad (36)
\]

In the above expressions (33), (35) and (37) are derived for approach process and (34), (36) and (38) are derived for recess process. Axial compressive stiffness will remain same and its values are given in (26) and (27).

3.3.2. Effect of both misalignment and friction on TVMS of gear pair with cracked tooth

Bending stiffness;

\[
\frac{1}{k_{b,\text{crack}}} = \int_{\alpha_1}^{\alpha_2} \frac{12 \left[ 1 + \cos \alpha \cos \theta \left( \alpha_2 - \alpha \right) \sin \alpha - \cos \alpha \right] \cos \alpha \sin \alpha \right\}^2 \frac{dx}{(\alpha_2 - \alpha) \sin \alpha}, \quad (39)
\]

Shear stiffness;

\[
\frac{1}{k_{s,\text{crack}}} = \int_{\alpha_1}^{\alpha_2} \frac{2 \left[ 1 + \cos \alpha \cos \theta \left( \alpha_2 - \alpha \right) \sin \alpha - \cos \alpha \right] \cos \alpha \sin \alpha \right\}^2 \frac{dx}{(\alpha_2 - \alpha) \sin \alpha}, \quad (40)
\]

Torsion stiffness;

\[
\frac{1}{k_{t,\text{crack}}} = \int_{\alpha_1}^{\alpha_2} \frac{2 \left[ 1 + \cos \alpha \cos \theta \left( \alpha_2 - \alpha \right) \sin \alpha - \cos \alpha \right] \cos \alpha \sin \alpha \right\}^2 \frac{dx}{(\alpha_2 - \alpha) \sin \alpha}, \quad (41)
\]

In the above expressions (39), (41) and (43) are derived for approach process and (40), (42) and (44) are derived for recess process. Axial compressive stiffness will remain same and its values are given in (26) and (27).
4. Computer simulation of time-varying mesh stiffness

The main parameters of gear pair used in the computer simulation for calculation of TVMS are given in Table 3. The TVMS has been calculated using equations derived in Sections 3.1, 3.2 and 3.3

4.1. Time-varying mesh stiffness (TVMS) for misaligned shaft

The TVMS of gear pair mounted on aligned and misaligned shaft have been simulated using equations derived in Section 3.1. Angle of misalignment \( \theta \) is taken as 1°. Fig. 6 shows the comparison of TVMS for aligned and misaligned shaft for healthy gear tooth. A change in stiffness value is observed when misaligned shaft is considered, due to which stiffness value decreases. Fig. 7 shows the comparison of TVMS for aligned and misaligned shaft for cracked gear tooth. In Fig. 7, a reduction in stiffness is also observed due to cracked tooth.

Fig. 8 shows the percentage change in TVMS due to misalignment. It is clear from Fig. 8(a) and (b) that changes in TVMS is more during double tooth contact than the single tooth contact. Fig. 8(b) also suggests that during engagement of cracked tooth change in stiffness value decreases as compared with healthy tooth case.

4.2. Time-varying mesh stiffness including friction

The TVMS of spur gear pair has been simulated using equations derived in Section 3.2. Coefficient of friction value has been assumed as \( \mu = 0.1 \). The main parameters used in the time-varying stiffness calculation for the pinion and the gear are listed in Table 3. Fig. 9 shows the TVMS values for healthy gear pair for two cases viz. frictionless and considering friction. Fig. 10 shows the TVMS values for cracked gear tooth. A change in stiffness value is observed when friction is considered. Due to friction the value of TVMS varies. It has been observed that at the start of double point contact change in stiffness value is more which keeps on decreasing as the double tooth pair contact meshing continues. This variation in change in mesh stiffness is expected because it has been discussed in Section 3.2 that friction forces are acting in two different directions for approach process and recess process. In the case of double tooth pair contact first tooth pair contact recess process takes place whereas in second tooth pair contact approach process takes place. This phenomenon is the reason for the variation in mesh stiffness value considering friction. In the case of single tooth pair contact effect of approach process and recess process is clearly visible where due to approach process drop in mesh stiffness value is less as compared to healthy case. Also at the pitch point where friction forces change their direction a sudden drop in mesh stiffness value is noticed. This variation in mesh stiffness will increase the vibrations during gear meshing.

Fig. 11 shows the percentage change in TVMS due to friction force. Fig. 11(a) and (b) suggest that change in TVMS is positive during double tooth pair contact. A sudden change in TVMS value is observed during single tooth pair contact at pitch point. Also a sudden drop in stiffness value is observed when tooth contact changes from double pair contact to single pair contact.

4.3. TVMS with both friction and misalignment

The TVMS of spur gear pair with both friction and misalignment has been simulated using equations derived in Section 3.3. Angle of misalignment \( \theta \) is taken as 1° and coefficient of friction \( \mu \) is assumed as 0.1. Figs. 12 and 13 show the comparison of TVMS with and without consideration of both friction and misalignment during spur gear pair for healthy gear tooth and cracked gear tooth respectively. A change in stiffness value is observed in both cases.

Fig. 14 shows the percentage change in TVMS due to both misalignment and friction forces. It suggests that percentage change in TVMS is positive during both single and double tooth pair contact which shows that change in sign of percentage error as observed in Fig. 11 due to friction is overcome due to consideration of misalignment of shaft. A sudden change in TVMS value is observed during single tooth pair contact at pitch point.

The mesh stiffness of spur gear pair with and without crack is calculated as explained previously for one revolution of pinion angle. Table 4 shows the percentage reduction in mesh stiffness for healthy and cracked tooth against the pinion rotation angle for all the three cases as discussed in Section 3. Maximum and minimum percentage change in TVMS are 8.8254%
Fig. 6. Effect of misalignment on TVMS for healthy gear tooth.

Fig. 7. Effect of misalignment on TVMS of cracked gear tooth.

Fig. 8. Percentage change in TVMS due to misalignment (a) for healthy tooth, and (b) for cracked tooth.
**Fig. 9.** Effect of friction on TVMS of healthy gear.

**Fig. 10.** Effect of friction on TVMS of cracked gear tooth.

**Fig. 11.** Percentage change in TVMS due to friction (a) for healthy tooth, and (b) for cracked tooth.
Fig. 12. Effect of friction and misalignment on TVMS of healthy gear.

Fig. 13. Effect of friction and misalignment on TVMS of cracked gear tooth.

Fig. 14. Percentage change in TVMS due to both misalignment and friction forces; (a) for healthy tooth and (b) for cracked tooth.
and 0.3479% respectively which is observed when misalignment and friction forces both are considered in healthy spur gear pair mesh.

5. Conclusion

Based on model proposed in Refs. [2,3], in this study effect of shaft misalignment and friction forces have been explored. Despite the complexity in gear geometry, formulas for modeling of these newly included factors are all analytically derived. It presents a computer simulation based approach to study the effect of shaft misalignment and friction on the total effective mesh stiffness for spur gear pairs. An analytical model is developed by modifying the existing potential energy model for calculating the total mesh stiffness by including new term torsion stiffness in existing potential energy model. This study is further extended to study the effect of shaft misalignment and friction on TVMS of cracked tooth.

It has been found that the values of total effective mesh stiffness have changed as compared to healthy gear conditions. The mesh stiffness decreases due to misalignment of gear shaft and friction forces. Effect of friction during approach and recess is visible while calculating the TVMS. These results show that misalignment and friction cannot be neglected while calculating TVMS of gear pair. The TVMS values obtained from this study can be used in finding response of gear pair using dynamic modeling.

In future research emphasis will be given to investigate the effect of fillet foundation deflection and profile modifications on gear mesh stiffness. The results can be further modified for different gear parameters like gear tooth and rim configurations, change in pressure angle, crack initiation angles, manufacturing errors etc. in future.

References